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# The Theory Of Adaptive Elasticity (Hegedus And Cowin,1976) That Deals With Internal Bone Remodeling, Could Also Be Used In Order To Describe The Sur- Face Bone Remodeling.

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**Abstract**

In the present paper we proved that the theory of the adaptive elasticity (Hegedus and Cowin, 1976) that deals with internal bone remodeling, can also be used in order to study the surface bone remodeling. Particularly we considered the problem of a long bone which is under an axial load. Our theoretical findings, predicts the results of the studies that describes the athrophy (Uhthoff and Jaworski, 1978; Jaworski , et., al.,1980) and the hypertrophy of the bone ( Woo, et., al., 1981; Clisouras, 1984; Kaplan, 1997, Monaco, 1997, Beck, 1998; Amendola, 1999,Walker,1999; Bouche,1999; Coutoure and Karlson, 2002; Magnus-son, 2003, Hester, 2006, American Academy of Orthopaedic Surgeons, 2007) and comes to agreement with the classic theory of surface bo-ne remodeling, proposed by Cowin and Firoozbaksh (1981).

## INTRODUCTION

Living bone is continually undergoing processes of growth, reinforcement and resorption, termed collectively remodeling. There are two kinds of bone remodeling: internal and surface (Frost, 1964). Hegedus and Cowin (1976) proposed a theory for internal remodeling, ter-med as "theory of adaptive elasticity" which has been used in various problems (Cowin and Van-Buskirk,1978; Tsili, 2000; Qin and Ye, 2004).

The purpose of this work is to show that the theory of adaptive elasticity, can also be successfully used in order to study the surface remodeling of long bone.

## THE METHOD

Initially, that is for t <0, the long bone was under a steady state at which no remodeling occurred, subjected only to a constant compressive load $G_o$, due to vertical component of the ground reaction force at la-te stance phase, during normal walking. At t =0 the bo- ne is under a new compressive load G. We want to predict its surface remodeling, after a long time.

We model the long bone as a hollow circular cylinder, with an inner and outer radii a and b respectively. These radii are not constant, but they are altering du-ring the time, that is a = a(t) and b = b(t) with b(t) >a(t) The diaphyseal cross-section area S(t) is given by : $S(t) = \pi(b(t)^2 - a(t)^2) > 0$. The inner and outer radius and the cross-section area in reference configuration, were $a_o$, $b_o$ and $S_o = \pi(b_o^2 - a_o^2) > 0$ respectively.

The equations of the adaptive elasticity (Hegedus & Cowin, 1976) in cylindrical coordinates are, the rate re-modeling equation:

**Figure 1**

$$\dot{e}(t) = A(e) + A_T(e)[E_{rr} + E_{\theta\theta}] + A_Z(e)E_{zz} \qquad (1)$$

the strain- displacement equations:

**Figure 2**

$$E_{rr} = u_{r,r} \quad E_{\theta\theta} = (u_{\theta,\theta} + u_r)/r \quad E_{zz} = u_{z,z} \quad E_{r\theta} = (u_{\theta,z} - u_\theta/r)/2$$

$$E_{zr} = (u_{z,r} + u_{r,z})/2 \quad \text{and} \quad E_{\theta z} = (u_{z,\theta} + u_{\theta,z})/2 \qquad (2)_{1\text{-}2\text{-}3\text{-}4\text{-}5\text{-}6}$$

the stress in equilibrium state:

**Figure 3**

$$T_{rr,r} + T_{r\theta,\theta}/r + T_{rz,z} + (T_{rr} - T_{\theta\theta})/r = 0$$

$$T_{r\theta,r} + T_{\theta\theta,\theta}/r + T_{\theta z,z} + 2/r\,T_{r\theta} = 0$$

$$T_{rz,r} + T_{\theta z,\theta}/r + T_{zz,z} + T_{rz}/r = 0 \qquad (3)_{1\text{-}2\text{-}3}$$

The stress-strain relations for a transversely isotropic elastic material are:

**Figure 4**

$$T_{rr} = (\lambda_2 + 2\mu_2)E_{rr} + \lambda_2 E_{\theta\theta} + \lambda_1 E_{zz}$$

$$T_{\theta\theta} = \lambda_2 E_{rr} + (\lambda_2 + 2\mu_2)E_{\theta\theta} + \lambda_1 E_{zz}$$

$$T_{zz} = \lambda_1(E_{rr} + E_{\theta\theta}) + (\lambda_1 + 2\mu_1)E_{ZZ} \quad T_{r\theta} = 2\mu_2 E_{r\theta} \quad T_{rz} = 2\mu_A E_{rz} \quad T_{\theta Z} = 2\mu_A E_{\theta Z}$$

$(4)_{1\text{-}2\text{-}3\text{-}4\text{-}5\text{-}6}$

where:

**Figure 5**

$$\lambda_1 = \frac{v_A E_A E_T}{(1 - v_T)E_A - 2v_A{}^2 E_T} \qquad 2\mu_1 = \frac{(1 - v_T)E_A{}^2 - v_A E_A E_T}{(1 - v_T)E_A - 2v_A{}^2 E_T}$$

$$\lambda_2 = \frac{v_T E_T E_A + v_A^2 E_T^2}{(1 - v_T^2)E_A - 2v_A^2 E_T\left(1 + v_T\right)} \qquad 2\mu_2 = \frac{E_T}{1 + E_T} \qquad (5)_{1\text{-}2\text{-}3\text{-}4}$$

In rate remodeling equation (1), e is the change of the volume of the bone from its reference configuration, while A(e), $A_T$ (e), $A_A$ (e) are material coefficients, that is:

**Figure 6**

$$e(t) = \xi(t) - \xi_o \qquad (6)$$

where $\xi(t)$, $\xi_o$ are respectively the new and the initial volume. Since surface remodeling has to do with the deposition or resorption of bone mass to or from the endosteal and periosteal surface :

**Figure 7**

$$\xi(t) = L.S(t) = L\pi[b(t)^2 - a(t)^2] \qquad (7)$$

where L is the length of the long bone, which of course is contant . Then (6) is written as:

**Figure 8**

$$e(t) = L(S(t) - S_o) \qquad (8)$$

The rate remodeling equation (1) because of (8) takes the form:

**Figure 9**

$$dS(t)/dt = B(S) + B_T(S)[E_{rr} + E_{\theta\theta}] + B_Z(S)E_{zz} \qquad (9)$$

where B(S), $B_T$ (S), $B_Z$ (S) are material coefficients where:

**Figure 10**

$$B(S) = A(L(S - S_o))/L = A(e)/L$$

$$B_T(S) = A_T(L(S - S_o)/L = A_T(e)/L \quad \text{and}$$

$$B_Z(S) = A_Z(L(S - So))/L = A_Z(e)/L \qquad (10)_{1\text{-}2\text{-}3}$$

## THE PROBLEM AND THE SOLUTION

For t>0 , the only non vanishing stress is:

**Figure 11**

$$T_{Zz}(t) = -\frac{G}{S(t)} = \frac{-G}{\pi\left(b(t)^2 - a(t)^2\right)} \qquad (11)$$

while:

**Figure 12**

$$T_{rr}(t) = T_{r\theta}(t) = T_{rz}(t) = T_{\theta\theta}(t) = T_{\theta z}(t) = 0 \qquad (12).$$

The assumed solutions of our problem are:

**Figure 13**

$$u_r = A(t)r \quad u_\theta = 0 \quad \text{and} \quad u_z = B(t) = 0 \qquad (13)_{1\text{-}2\text{-}3}$$

where A(t), B(t) are unknown functions. Then (2) becau-se of (13) become:

**Figure 14**

$$E_{rr} = A(t) \quad E_{\theta\theta} = A(t) \quad E_{zz} = B(t) \quad E_{rz} = E_{r\theta} = E_{z\theta} = 0 \quad (14)$$

Then (4) because of (11), (12) and (14) become:

**Figure 15**

$$2(\lambda_2 + \mu_2)A(t) + \lambda_1 B(t) = 0 \quad \text{and}$$

$$2\lambda_1 A(t) + (\lambda_1 + 2\mu_1)B(t) = -G/S(t)\pi \qquad (15)_{1\text{-}2}$$

Therefore:

**Figure 16**

$$A(t) = -\lambda_1 G / 2\pi S(t) f \qquad B(t) = -(\lambda_2+\mu_2) G / \pi S(t) f \qquad (16)_{1\text{-}2}$$

where:

**Figure 17**

$$f = (\lambda_2 + \mu_2)(\lambda_1 + 2\mu_1) - 2\lambda_1^2 \qquad (17)$$

Substituting (16) and (17) into (2) we find the strains.

Then the remodeling equation (9) becomes:

**Figure 18**

$$dS/dt = B(S) -$$

$$([B]_T(S)\lambda_1(S) + B_Z(S)(\lambda_2(S) + \mu_2(S))G)/\pi S(t) f$$

(18)

We assume that the movements of endosteal and periosteal surfaces are small. The last results that the change of the cross- section area S(t), is also small. For that reason we use the following approximate li-near forms for the material coefficients:

**Figure 19**

$$B(S) = b_1 S + b_o \qquad B_T(S) = b_T S + b_T \qquad B_z(S) = b_z S + b_z$$

$$\lambda_1(S) = \Lambda_1 + \Lambda_1 S \qquad \lambda_2(S) = \Lambda_2 + \Lambda_2 S \qquad \mu_1(S) = M_1 + M_1 S$$

$$\mu_2(S) = M_2 + M_2 S \qquad (19)_{1\text{-}2\text{-}3\text{-}4\text{-}5\text{-}6\text{-}7}$$

Consequently (18) by the help of (19) concludes:

**Figure 20**

$$dS(t)/dt = b_1 S + b_o - \frac{[[b]_T \Lambda_1 + b_z(\Lambda_2 + M_2)]G}{\pi S F} \qquad (20)$$

where:

**Figure 21**

$$F = (\Lambda_2 + M_2)(\Lambda_1 + 2M_1) - 2\Lambda_1^2 \qquad (21)$$

that is we conclude to an equation of the form:

**Figure 22**

$$dS/dt = [b_1 S^2 + b_o S + c_o]/S \qquad (22)$$

with the initial condition:

**Figure 23**

$$S(0) = S_o \qquad (23)$$

In order the solutions to have physical sense, they

must be positive, that is for $t \to +\infty$, $\lim S(t) > 0$.

## THE SOLUTIONS AND THE PHYSICAL MEANING

The solution of (22) that satisfies (23) is :

**Figure 24**

$$b_1 S(t)^2 + b_o S(t) + c_o = e^{2b_1 t}(b_1 S_o^2 + b_o S_o + c_o) \qquad (24)$$

We define $\Delta = b_o^2 - 4b_1 c_o$ and we distinguish the following cases: 1) $\Delta > 0$. Then (24) takes the following form:

**Figure 25**

$$[S(t) + b_o/2b_1]^2 = [e^{2b_1 t}(b_1 S_o^2 + b_o S + c_o) + (\sqrt{\Delta}/2b_1)^2]/b_1 \qquad (25)$$

a)If $b_1 > 0$, then for $t \to +\infty$, it follows $S(t) \to \infty$. This solution is rejected, because it is unstable. b) If $b_1 < 0$, then there is no solution, since the limit of S(t) for $t \to +\infty$ does not exist. c) Finally if $b_1 = 0$, then the solution is:

**Figure 26**

$$S(t) = e^{b_o t}[S_o b_o + c_o] - c_o/b_o \qquad (26)$$

We distinguish the following subcases: i) If $b_o > 0$, then for $t \to +\infty$, it results that $S(t) \to +\infty$ and this solution is rejected .ii) If $b_o < 0$, for $t \to +\infty$, it implies that $S(t) \to -c_o/b_o$. If $-c_o/b_o \leq 0$, then the solution is rejected. If $-c_o/b_o > 0$, then the solution is accepted. The cases $-c_o/b_o > S_o$, $-c_o/b_o < S_o$ and $-c_o/b_o = S_o$ correspond to hypertrophy, atrophy and steady state of the bone respectively. iii) If $b_o = 0$, then the solution reduces to the form:

**Figure 27**

$$S(t)^2 = 2c_o t + S_o^2 \qquad (27)$$

If $c_o > 0$, then for $t \to +\infty$ ,it follows that $S(t) \to \infty$. If $c_o < 0$, the limit does not exist. Finally if $c_o = 0$, then for $t \to +\infty$ it follows that $S(t) \to S_o$. This solution is accepted and states that after a long time, the bone will continue to be in the same steady state.

2) If $\Delta = 0$, then (25) reduces to the following form:

**Figure 28**

$$[S(t)+ b_o / 2b_1]^2 = e^{2b_1t}[b_1S_o^2 + b_oS_o + c_o]/ b_1 \qquad (28)$$

a) If $b_1 > 0$, then for t□+∞, it follows that S(t) □∞. b) If $b_1 < 0$, then for t□+∞ it results that S(t) □ □$b_o / 2b_1$. If $b_o \leq 0$, then □$b_o / 2b_1 \leq 0$ and this solution is rejected. Finally if $b_o > 0$, then S(t) >0 since □$b_o/2b_1 > 0$. This solution is accepted. The cases: □$b_o/ 2b_1 > S_o$, □$b_o/2b_1 < S_o$ and □$b_o / 2b_1 = S_o$ correspond to hypertrophy, atrophy and steady state of the bone respectively. 3) If □ < 0, then (24) takes the following form:

**Figure 29**

$$[S(t) + b_o /2b_1]^2 = [e^{2b_1t}(b_1S_o^2 + b_oS_o + c_o) - (\sqrt{-\Delta} /2b_1)^2]/b_1$$

$$(29)$$

i)If $b_1 > 0$ then for t□+ ∞ it follows S(t)□∞, that is the solution is rejected. ii) If $b_1 = 0$, then we result to equa- tion (26). iii) Finally if $b_1 < 0$, then for t□ +∞, it follows that S(t)□k, or S(t) □ □ k , where:

**Figure 30**

$$k = [\sqrt{(-\Delta/ 4b_1^3)} - b_o/2b_1] \qquad (30)$$

If k > 0, then the first solution is accepted, while second is rejected. In addition the cases: $k > S_o$, $k < S_o$ and $k = S_o$ correspond to hypertrophy, atrophy and state steady of the bone. If □k <0,then the first solution is rejected, while second is accepted. Moreover the cases □$k> S_o$, □$k < S_o$ and □$k =S_o$ correspond to hypertrophy, atrophy and steady state of the bone.

## DISCUSSION

Our model predicts the results of previous studies that describes the hypertrophy (Woo, et., al.,1981) and the atrophy of bone ( Uhthoff and Jaworski 1979; Jaworski et., al., 1980). These studies are also cited in the classic theory of surface bone remodeling, proposed by Cowin and Firoozbakhsh (1981).

In addition concerning the tibia, the model predicts a pathological case of this bone, termed as “ Medial Tibial Stress Syndrome” or “ Shin Splints”. This case is characterized by a periosteal inflammation of the bone and a narrow bone density (Clisouras,1984; Mona-co et. ,al., 1997; Kaplan et., al.,1997; Beck,1998; Amendola et., al., 1999; Bouche et., al., 1999; Walker, 1999; Couture and Karlson, 2002; Magnusson et., al., 2003; Romansky and Erfle, 2003; Hester, 2006; Academy of Orthopaedic Surgeons, 2007) resulting to tibia’s hypertrophy. Therefore, our problem has three possible solutions. After a long time, the long bone will increase or will decrease its cross section area, or it will be in a steady state if $G = G_o$

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